- You have to sign all sheets with your name and your student number.
- You may use only the paper provided by the organiser.
- Use of any electronic devices is strictly prohibited.
- Each problem is worth 10 points.
- You should solve *exactly* 4 out of the 5 problems given below.
  In case you submit 5 solutions, only 4 will be evaluated (at random).
- You can use any fact that was proven in the lecture or in class.

Problem 1. Let $A \subseteq \mathbf{R}^2$ be $\mathscr{L}^2$-measurable with $\mathscr{L}^2(A) > 0$. Show that A contains vertices of some regular hexagon.

Hint. Balls are rotationally invariant.

Problem 2. We say that $f : \mathbf{R} \to \mathbf{R}$ is *good* if:

> for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any $n \in \mathscr{P}$ if $-\infty < x_1 \leqslant y_1 \leqslant \cdots \leqslant x_n \leqslant y_n < \infty$ satisfy $\sum_{i=1}^n |x_i - y_i| \leqslant \delta$, then $\sum_{i=1}^n |f(x_i) - f(y_i)| \leqslant \varepsilon$.

Assume f is good and non-decreasing. Let ψ be the associated Riemann-Stieltjes measure. Show that $\psi \ll \mathscr{L}^1$.

Problem 3. Assume $0 < C < \infty$, $k, n \in \mathscr{P}$, $k < n$,

$$\begin{gathered} \phi, \phi_1, \phi_2, \ldots \quad \text{are non-zero Radon measures over } \mathbf{R}^n\,, \\ \operatorname{spt}\phi_i \text{ is path connected for } i \in \mathscr{P}\,, \quad \operatorname{spt}\phi \text{ is compact}\,, \\ \phi_i \to \phi \quad \text{as } i \to \infty \text{ (weakly)}\,, \\ \text{and} \quad \phi_i(\mathbf{B}(x,r)) \geqslant C r^k \quad \text{whenever } 0 < r < 1,\ i \in \mathscr{P}, \text{ and } x \in \operatorname{spt}\phi_i\,. \end{gathered}$$

Show that for any $\delta > 0$ there exists $i_0 \in \mathscr{P}$ such for all $\mathscr{P} \ni i > i_0$

$$\operatorname{spt}\phi_i \subseteq \Big(\operatorname{spt}\phi + \mathbf{B}(0,\delta)\Big) = \mathbf{R}^n \cap \Big\{x : \operatorname{dist}(x, \operatorname{spt}\phi) \leqslant \delta\Big\}\,.$$

Recall. If ψ is a Radon measure over $\mathbf{R}^n$, then $\operatorname{spt}\psi = \mathbf{R}^n \sim \bigcup\{V : \psi(V) = 0,\ V \text{ is open in } \mathbf{R}^n\}$.

Problem 4. Let $A \subseteq \mathbf{R}^n$ be arbitrary and ϕ be a Radon measure over $\mathbf{R}^n$. Show that $\mathbf{D}(\phi \llcorner A, \phi, x) = 1$ for ϕ almost all $x \in A$.

Remark. This was not proven in the class.

Problem 5. Let $0 < d < \infty$, $0 < \gamma < 1$, $1 \leqslant C < \infty$, (X,ρ), (Y,σ) be metric spaces, and for $0 \leqslant s < \infty$ let $\mathscr{H}^s_X$ and $\mathscr{H}^s_Y$ be the s-dimensional Hausdorff measures over X and Y respectively. Assume $f : X \to Y$ satisfies $\sigma(f(x), f(y)) \leqslant C\rho(x,y)^\gamma$ whenever $x, y \in X$. Show that

$$\mathscr{H}^d_Y(f[A]) \leqslant \frac{C^d \boldsymbol{\alpha}(d)}{\boldsymbol{\alpha}(d\gamma) 2^{d(1-\gamma)}} \mathscr{H}^{d\gamma}_X(A) \quad \text{whenever } A \subseteq X\,.$$

Conclude that $\dim_{\mathscr{H}} f[A] \leqslant \gamma^{-1} \dim_{\mathscr{H}} A$.

Solution of problem 1.
Recall that $\mathbf{D}(\mathscr{L}^2 \llcorner A, \mathscr{L}^2, x) = 1$ for $\mathscr{L}^2$ almost all $x \in A$. Let $x \in A$ be such a point and assume $x = 0$. Let $\rho_i : \mathbf{R}^2 \to \mathbf{R}^2$ be the clockwise rotation by angle $2\pi i/6$ for $i \in \{1, 2, \dots, 6\}$. Observe that $\mathscr{L}^2(A \cap \mathbf{B}(0,r)) = \mathscr{L}^2(\rho_i[A] \cap \mathbf{B}(0,r))$ for $r > 0$ and $i \in \{1, 2, \dots, 6\}$; hence, $\mathbf{D}(\mathscr{L}^2 \llcorner \rho_i[A], \mathscr{L}^2, x) = 1$. Choose $r > 0$ such that

$$\mathscr{L}^2(\rho_i[A] \cap \mathbf{B}(0,r)) > 5/6 \mathscr{L}^2(\mathbf{B}(0,r)) \quad \text{for } i \in \{1, 2, \dots, 6\},$$

then

$$\mathscr{L}^2\Big(\textstyle\bigcap_{i=1}^6 \rho_i[A] \cap \mathbf{B}(0,r)\Big) > 0$$

so there exists $z \in \bigcap_{i=1}^6 \rho_i[A]$ which means that $\rho_i^{-1}(z) \in A$ for $i \in \{1, 2, \dots, 6\}$. The six points $\{\rho_i^{-1}(z) : i \in \{1, 2, \dots, 6\}\}$ are vertices of a regular hexagon.

Solution of problem 2.
Recall the definition of ψ. For $0 < \delta \leqslant \infty$ and $A \subseteq \mathbf{R}$ we have

$$\psi_\delta(A) = \inf \left\{ \sum_{I \in F} \operatorname{diam} f[I] : \begin{array}{c} F \text{ is countable family of open intervals,} \\ A \subseteq \bigcup F, \quad \operatorname{diam} I \leqslant \delta \ \text{ for } I \in F \end{array} \right\}$$
$$\text{and} \quad \psi(A) = \sup_{\delta > 0} \psi_\delta(A) = \lim_{\delta \downarrow 0} \psi_\delta(A).$$

Note that being *good* is stronger than being *continuous* so f is continuous; in particular, it does not have any jumps so $\operatorname{diam} f[I] = \operatorname{diam} f[\operatorname{Clos} I]$. Therefore, the measure ψ will not change if we allow F inside the infimum to contain also closed and half-closed intervals.
Let $Z \subseteq \mathbf{R}$ be such that $\mathscr{L}^1(Z) = 0$. Let $\varepsilon > 0$ and choose $\delta > 0$ for f as in the definition of good functions. Since $\mathscr{L}^1(Z) = 0$ for any $0 < \iota < \delta$ there exists a countable family F of open intervals such that

$$Z \subseteq \textstyle\bigcup F \quad \text{and} \quad \sum_{J \in F} \operatorname{diam} J \leqslant \iota < \delta.$$

Let us order $F = \{J_1, J_2, \dots\}$ and set $A_j = J_j \sim \bigcup_{i=1}^{j-1} J_i$ for $j \in \mathscr{P}$. Clearly $\{A_j : j \in \mathscr{P}\}$ is disjointed and each A_j is a sum of a finite number of pairwise disjoint intervals (open, closed, or half-closed). Let G be the family of all intervals of all A_j for $j \in \mathscr{P}$, then G is a countable disjointed family of intervals such that

$$Z \subseteq \textstyle\bigcup G \quad \text{and} \quad \sum_{J \in G} \operatorname{diam} J \leqslant \sum_{J \in F} \operatorname{diam} J \leqslant \iota < \delta$$

For $J \in G$ set $a_J = \inf J$ and $b_J = \sup J$. Assume $\inf Z > M > -\infty$ so that we can require that $a_J \geqslant M$ for $J \in F$. Let us order the set of pairs $\{(a_J, b_J) : J \in G\} = \{(x_i, y_i) : i \in \mathscr{P}\}$ so that $x_i \leqslant x_{i+1}$ for $i \in \mathscr{P}$. Since G is disjointed we also have $y_i \leqslant x_{i+1}$ for $i \in \mathscr{P}$. For each $n \in \mathscr{P}$ we have, due to our choice of δ,

$$\textstyle\sum_{i=1}^n |x_i - y_i| \leqslant \sum_{J \in G} \operatorname{diam} J \leqslant \iota < \delta \quad \text{and} \quad \sum_{i=1}^n |f(x_i) - f(y_i)| \leqslant \varepsilon.$$

Passing to the limit $n \to \infty$ we obtain

$$\textstyle\sum_{J \in G} \operatorname{diam} f[J] = \sum_{i=1}^\infty |f(x_i) - f(y_i)| \leqslant \varepsilon.$$

Clearly $\operatorname{diam} J \leqslant \iota$ for $J \in G$ and we get

$$\psi_\iota(Z) \leqslant \varepsilon.$$

Letting first $\iota \downarrow 0$ and then $\varepsilon \downarrow 0$ we get $\psi(Z) = 0$. In case $\inf Z = -\infty$ we consider an increasing sequence $Z_k = Z \cap \{t : t > -k\}$ for $k \in \mathscr{P}$ and argue that each Z_k is a ψ null set so it is ψ-measurable and we can write

$$\psi(Z) = \psi(\bigcup\{Z_k : k \in \mathscr{P}\}) = \lim_{k\to\infty} \psi(Z_k) = 0\,.$$

Solution of problem 3.

First we shall prove that for each $\delta > 0$ there exists $i_0 \in \mathscr{P}$ such that for $i \in \mathscr{P}$ with $i \geqslant i_0$ there exists $y_i \in \operatorname{spt}\phi_i$ such that $\operatorname{dist}(y_i, \operatorname{spt}\phi) \leqslant \delta$. Assume the contrary, i.e., that $\rho_i = \inf\{|x-y| : x \in \operatorname{spt}\phi,\, y \in \operatorname{spt}\phi_i\} > \delta$ for all $i \in \mathscr{P}$ larger than some $i_0 = i_0(\delta) \in \mathscr{P}$ depending on δ. Let $f \in \mathscr{K}(\mathbf{R}^n)$ be such that $f(z) = 1$ whenever $\operatorname{dist}(z, \operatorname{spt}\phi) \leqslant \delta/2$ and $f(z) = 0$ if $\operatorname{dist}(z, \operatorname{spt}\phi) \geqslant \delta$. Such f exists since $\operatorname{spt}\phi$ is compact and non-empty. Clearly

$$\lim_{i\to\infty} \textstyle\int f \,\mathrm{d}\phi_i = \lim_{i\to\infty} \int_{\operatorname{spt}\phi_i} 0 \,\mathrm{d}\phi_i = 0 \quad \text{but} \quad \int f\,\mathrm{d}\phi = \phi(\operatorname{spt}\phi) > 0 \quad \text{a contradiction}\,.$$

Now, we turn to the proof of the main claim. Assume the contrary, i.e., that there exists $\delta > 0$ such that for all $i \in \mathscr{P}$ there exists $x_i \in \operatorname{spt}\phi_i$ with $\operatorname{dist}(x_i, \operatorname{spt}\phi) > \delta$. If $i \geqslant i_0(\delta)$, then the first part of our proof above yields a point $y_i \in \operatorname{spt}\phi_i$ with $\operatorname{dist}(y_i, \operatorname{spt}\phi) \leqslant \delta$ and, since $\operatorname{spt}\phi_i$ is path connected, we may and shall assume that $\operatorname{dist}(x_i, \operatorname{spt}\phi) \leqslant 2\delta$. Since $\operatorname{spt}\phi$ is compact we see that $\operatorname{spt}\phi + \mathbf{B}(0, 2\delta)$ is also compact and we may assume (possibly choosing a sub-sequence) that $\lim_{i\to\infty} x_i = x$ exists. Clearly $\delta \leqslant \operatorname{dist}(x, \operatorname{spt}\phi) \leqslant 2\delta$. Let $f \in \mathscr{K}(\mathbf{R}^n)$ be such that $f \geqslant 0$, $f(z) = 1$ whenever $\delta/2 \leqslant \operatorname{dist}(z, \operatorname{spt}\phi) \leqslant 3\delta$ and $f(z) = 0$ for $z \in \operatorname{spt}\phi$. We have

$$\textstyle\int f\,\mathrm{d}\phi_i \geqslant \int_{\mathbf{B}(x,\delta/2)} f\,\mathrm{d}\phi_i \geqslant \int_{\mathbf{B}(x_i,\delta/2-|x-x_i|)} f\,\mathrm{d}\phi_i \geqslant \phi_i(\mathbf{B}(x_i, \delta/4)) \geqslant C4^{-k}\delta^k > 0$$

whenever $i \in \mathscr{P}$ is so big that $|x - x_i| \leqslant \delta/4$. Since $\int f\,\mathrm{d}\phi = 0$, this contradicts $\phi_i \to \phi$ as $i \to \infty$.

Solution of problem 4.

Observe that since A may be ϕ non-measurable the measure $\phi \llcorner A$ might not be Borel regular; hence, $\phi \llcorner A$ might not be Radon and we cannot directly use any results concerning existence of densities for Radon measures.

Let $K \subseteq \mathbf{R}^n$ be compact and $C = A \cap K$. We shall show that $\mathbf{D}(\phi \llcorner C, \phi, x) = 1$ for ϕ almost all $x \in C$. Since $\mathbf{R}^n$ is a sum of countably many compact sets, this will suffices to prove the main claim.

If $\phi(C) = 0$, then there is nothing to prove so assume $\phi(C) > 0$. Let $B \subseteq \mathbf{R}^n$ be Borel and such that $C \subseteq B$ and $\phi(B) = \phi(C)$. Then B is a ϕ-hull of C, i.e., $\phi(T \cap C) = \phi(T \cap B)$ for any ϕ-measurable set T. Since balls are ϕ measurable we get

$$\mathbf{D}(\phi \llcorner C, \phi, x) = \lim_{r\downarrow 0} \frac{\phi(C \cap \mathbf{B}(x,r))}{\phi(\mathbf{B}(x,r))} = \lim_{r\downarrow 0} \frac{\phi(B \cap \mathbf{B}(x,r))}{\phi(\mathbf{B}(x,r))} = \lim_{r\downarrow 0} \fint_{\mathbf{B}(x,r)} \mathbb{1}_B \,\mathrm{d}\phi = \mathbb{1}_B(x)$$

for ϕ almost all x. In particular, $\mathbf{D}(\phi \llcorner C, \phi, x) = 1$ for ϕ almost all $x \in B$ but $C \subseteq B$ so $\mathbf{D}(\phi \llcorner A, \phi, x) = 1$ for ϕ almost all $x \in C$.

Solution of problem 5.

Note that $\operatorname{diam} f[S] \leqslant C(\operatorname{diam} S)^\gamma$ whenever $S \subseteq X$. Let $A \subseteq X$. Employing the fact that $\inf C \leqslant \inf D$ whenever $D \subseteq C \subseteq \mathbf{R}$, for $0 < \delta \leqslant \infty$, setting $\kappa = (\frac{\delta}{C})^{1/\gamma}$, we get

$$
\begin{aligned}
&\frac{2^d}{\boldsymbol{\alpha}(d)} \mathscr{H}^d_\delta(f[A]) \\
&\quad = \inf \left\{ \sum_{P \in F} (\operatorname{diam} P)^d : F \subseteq \mathbf{2}^Y,\ F \text{ countable}\,,\ f[A] \subseteq \bigcup F,\ \operatorname{diam} P \leqslant \delta \text{ for } P \in F \right\} \\
&\quad \leqslant \inf \left\{ \sum_{S \in G} (\operatorname{diam} f[S])^d : G \subseteq \mathbf{2}^X,\ G \text{ countable}\,,\ A \subseteq \bigcup G,\ \operatorname{diam} S \leqslant \kappa \text{ for } S \in G \right\} \\
&\quad \leqslant \inf \left\{ \sum_{S \in G} C^d (\operatorname{diam} S)^{d\gamma} : G \subseteq \mathbf{2}^X,\ G \text{ countable}\,,\ A \subseteq \bigcup G,\ \operatorname{diam} S \leqslant \kappa \text{ for } S \in G \right\} \\
&\quad = \frac{C^d 2^{d\gamma}}{\boldsymbol{\alpha}(d\gamma)} \mathscr{H}^{d\gamma}_\kappa(A) \leqslant \sup_{\iota > 0} \frac{C^d 2^{d\gamma}}{\boldsymbol{\alpha}(d\gamma)} \mathscr{H}^{d\gamma}_\iota(A) = \frac{C^d 2^{d\gamma}}{\boldsymbol{\alpha}(d\gamma)} \mathscr{H}^{d\gamma}(A)\,.
\end{aligned}
$$

Passing to the limit $\delta \downarrow 0$, we get the claim.

If $\mathscr{H}^s(A) = 0$ for some $0 \leqslant s < \infty$, then $\mathscr{H}^{s/\gamma}(f[A]) = 0$; hence, if $\dim_{\mathscr{H}}(A) < s$, then $\dim_{\mathscr{H}} f[A] \leqslant s/\gamma$ so $\dim_{\mathscr{H}} f[A] \leqslant \dim_{\mathscr{H}}(A)/\gamma$.